Comment on 'Penetrability of a one-dimensional Coulomb potential' by M. Moshinsky

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COMMENT

Comment on 'Penetrability of a one-dimensional Coulomb potential' by M Moshinsky

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Abstract. The analysis of the one-dimensional Schrödinger equation with the potential $-\lambda/x$ given in [1] is found to be incorrect. Contrary to the author's claim, the origin is indeed 'impenetrable'.

In reference [1] Moshinsky discussed the bound state and scattering solutions of the one-dimensional Schrödinger equation with the potential $-\lambda/x$, i.e.

$$-\psi'' - \frac{\alpha}{x}\psi = k^2\psi \tag{1}$$

where $\alpha := 2m\lambda/\hbar^2$ and $k^2 := 2mE/\hbar^2$. As mentioned in [1] there is extensive literature [2–10] on the solutions of the equation

$$-\psi'' - \frac{\alpha}{|x|}\psi = k^2\psi. \tag{2}$$

There has also been considerable controversy, but the author ignores these discussions, which are equally applicable to (1), and tries to solve (1) without paying any attention to them.

In each of the regions $x > 0$ and $x < 0$ equations (1) and (2) have two linearly independent solutions: one of these is the 'regular' Whittaker function, which is analytic in the vicinity of $x = 0$, and the other is 'irregular,' with a logarithmic singularity such that its derivative at $x = 0$ is infinite. There are convincing arguments, given in [5] and [6], that the irregular solution is not acceptable for equation (2), and these arguments hold for (1) as well†. As pointed out by Andrews in [5], the fact that only one solution is acceptable on each half-line implies that at positive energies there can be no transmission into the other half, because the only acceptable solution that contains no incoming wave, as required for transmission, is the trivial one. This is why the origin is said to be 'impenetrable.' Therefore, for both (1) and (2) the reflection coefficients are equal to 1, contrary to the claim of [1]. The argument in [1], using equation (4.3) to match infinite derivatives, is ingenious but irrelevant. On what grounds, other than the fact that both sides are finite, is that supposed to be the correct matching condition? The continuity of a solution of a second-order differential equation and of its first derivative is dictated by the differential equation; (4.3) has no such justification.

† An additional argument is that if the irregular solution were acceptable, there would be no criterion for a spectrum for the positive half-line alone; in contrast to the case of a non-singular potential, when the spectrum depends on the boundary condition chosen at $x = 0$, here any addition of a multiple of the irregular function makes the derivative infinite, and a sensible boundary condition other than regularity does not exist.

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For negative energies, the acceptable solution on the right is the regular Whittaker function $M_{\beta,\frac{1}{2}}(2\kappa x)$, with $\beta := \alpha/2\kappa$ and $\kappa := \sqrt{-2mE/\hbar^2}$. For negative values of x we may take the function $-M_{-\beta,\frac{1}{2}}(2\kappa|x|)$ as a solution of (1) and $-M_{\beta,\frac{1}{2}}(2\kappa|x|)$, as a solution of (2). That gives us solutions of (1) and (2) that are continuous and have continuous first derivatives at $x = 0$; they are therefore solutions for $-\infty < x < \infty$. Since $M_{\beta,\frac{1}{2}}(2\kappa x)$ is exponentially decreasing at infinity for $\beta = n$, where n is a positive integer, the point spectrum of (2) is $\kappa^2 = \alpha^2/n^2$ if $\alpha > 0$. (For $\alpha \leqslant 0$ the point spectrum of (2) is, of course, empty.) However, as cogently argued in [5], [8] and [9], a step-discontinuity of the derivative at $x = 0$ is acceptable†. Therefore (1) has the same spectrum‡, with the support of the wave function entirely on the right for $\alpha > 0$ and on the left for $\alpha < 0$. That same argument allows, for each eigenvalue, two linearly independent solutions for (2), one supported on the left, the other on the right (or one even and the other odd). Thus the bound states of (2) are doubly degenerate, as already pointed out in [2] and also discussed in [7]. Equation (1) has no such degeneracy.

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† That argument was already implicitly assumed here for positive energies. Without it there would be no non-trivial positive-energy solutions.

‡ This agrees with the result of [1], although it was there obtained rather more indirectly.